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LINEAR SENSITIVITY ANALYSIS OF THE CONTINUOUS-TIME MATRIX LYAPUNOV INEQUALITY

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Abstract. In this paper we consider an approach to obtain admissible solution sets of the continuous-time Lyapunov matrix inequalities using the solution sets of the corresponding continuous Lyapunov equations, under some solvability conditions. We obtain tight perturbation bounds for the considered continuous-time Lyapunov matrix inequalities, which are linear functions of the data perturbations.Numerical examples are also presented.

Keywords: LMI admissible set, Lyapunov based criteria, Lyapunov equation, sensitivity analysis.

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1 Introduction

In many control problems the design constraints can be reformulated and solved with respect to Linear Matrix Inequalities (LMIs). This is a not surprising fact since LMIs are direct byproducts of Lyapunov based criteria, and that Lyapunov techniques play an essential role in the analysis, design and control of linear systems. An overview of the LMI application can be found in (Boyd et al., 1996; Scherer & Weiland, 2004/2005; Yonchev et al., 2005).

The asymptotic stability analysis of a linear autonomous system is in fact a good illustration of the LMIs usage in control theory. Based on the Lyapunov criteria such analysis can be performed by applying the Lyapunov based inequality. The aim of the paper is to investigate an approach to obtain an admissible solution set of the continuous-time matrix Lyapunov inequality by using the corresponding the solution sets of the continuous-time Lyapunov equation. Afterwards tight perturbation bounds of the admissible solutions of the considered LMI have to be computed (Scalcon et al., 2020; Deaecto & Daiha, 2020; Rauh et al., 2020).

Throughout the paper following notation is adopted: $R^{m\times n}$- the space of real $m\times n$ matrices; $R^n = R^{n\times 1}$; I_n - the identity $n\times n$ matrix; e_n- the unit $n\times 1$ vector; M^T- the transpose of M; $M^{\perp}$- the pseudo inverse of M ;$||M||_2 = \sigma_{\max}(M)$ - the spectral norm of M, where $\sigma_{\max}(M)$ is the maximum singular value of M; $vec(M) \in R^{m\,n}$- the column-wise vector representation of $M \in R^{m\times n}$; $\Pi_{m,n} \in R^{m\,n\times m\,n}$- the vec-permutation matrix, such that $vec(M^T) = \Pi_{m,n} vec(M)$; $M \otimes P$- the Kroneker product of the matrices M and P. The notation ":=" stands for "equal by definition", $S^+(n)$ is a space of positive definite matrices of size n.

The remaining part of the paper is formulated in the following way. In Section 2 we briefly present problem set up. In Section 3 we describe the proposed solution approach in order to obtain an admissible solution set of the matrix Lyapunov inequality and its sensitivity bounds. In Section 4 we present some numerical examples, the obtained results and discussions. The paper concludes in Section 5 with some final remarks.

2 Aim and problem formulation

The aim of the paper is to present an approach to obtain an admissible solution set of the continuous-time matrix Lyapunov inequality using the solution set of the corresponding continuous-time Lyapunov equation. Also the problem formulation includes computation of linear perturbation bounds for the set of admissible solutions of the continuous-time matrix Lyapunov inequality.

Let us consider the following linear continuous-time autonomous system

$$\dot{x}(t) = Ax(t), A \in R^{n\times n} \tag{1}$$

here $x(t) \in R^n$ is the system state and A, and B are constant matrices of corresponding size.

Definition 1. *(LMI) (Scherer & Weiland, 2004/2005; Yonchev et al., 2005). Linear matrix inequality is an expression of the type*

$$F(x) := F_0 + x_1F_1 + \cdots + x_mF_m > 0, \tag{2}$$

where
1. $x = (x_1, x_2, \ldots, x_m)$ is a vector of real numbers;
2. $F_0, F_1, \ldots, F_m$ are real symmetric matrices, i.e. $F_i = F_i{}^T \in R^{n\times n}, i = 0, 1, \ldots, m$;
3. the inequality $F(x) > 0$ in expression (2) means, that the matrix $F(x)$ is positive definite, i.e. $u^TF(x)u > 0$ for each $0 \neq u \in R^n$.

There exist three main problem connected with studying and application of the LMIs (Scherer & Weiland, 2004/2005; Yonchev et al., 2005).

1. **Admissibility of LMIs**. To check if there exist solutions x of the LMI $F(x) > 0$, is called *admissibility problem.*
2. **Optimization with LMI constraints.**
3. **Generalized eigen value problem.**

More specifically we are interested in the first problem. The autonomus system (1) is called asymptotically stable if and only if there exists a symmetric positive definite $n \times n$ matrix X, such that:

$$A^TX + XA < 0 \tag{3}$$

This result is obtained by A. Lyapunov. With other words it is necessary to find an admissible solution $X_0 = X_0{}^T > 0$ of the LMI

$$\begin{bmatrix} -X_0 & 0 \\ 0 & A^TX_0 + X_0A \end{bmatrix} < 0$$

The main aim of the publication is connected with performing a sensitivity analysis of the matrix Lyapunov inequality (3) for a stable matrix A. Sensitivity analysis of the matrix Lyapunov equation is performed in (Konstantinov et al., 1995; Konstantinov et al., 1999).

We consider the following perturbations ΔA, which are applied to the matrix A. The the following perturbed LMI can be obtained:

$$(A + \Delta A)^TX + X(A + \Delta A) < 0. \tag{4}$$

Further we assume that the perturbations ΔA do not change the sign of the LMI (4). With $X_P = X_0 + \Delta X_0$ we denote an admissible solution of the LMI (4), having in mind that X_0 is an admissible solution of the LMI (3).

To perform a linear sensitivity analysis it is necessary that we obtain an analytical description of the admissible solution sets of the LMIs (3) and (4) of the following type:

$$\Omega := \{X_0 \in S^+(n) : A^TX_0 + X_0A < 0\} \tag{5}$$

and

$$\Omega_P := \{X_P \in S^+(n) : (A + \Delta A)^TX_P + X_P(A + \Delta A) < 0\} \tag{6}$$

respectively.

3 Solution approach

We consider the following Lyapunov equation:

$$A^T X + XA = -Q, Q \in S^+(n). \tag{7}$$

A single solution of the equation (7) for a particular matrix $Q_0 \in S^+(n)$ is shown:

$$X_0 = f(A, Q_0) \in S^+(n) \tag{8}$$

Then the solution set for the expression (7) is given by $\Psi := \{f(A,Q) \in S^+(n) : (7)\}$. An approach to compute an admissible solution set for the LMI (3) is to use the solution set of the Lyapunov equation on condition that $Q_0, Q \in S^+(n)$ is fulfilled, i.e.:

$$\Omega := \{f(A,Q) : Q \in S^+(n)\} \tag{9}$$

We claim that the perturbed admissible solution X_P of the LMI (4) is a solution of the Lyapunov equation with slightly perturbed part, i.e.:

$$(A + \Delta A)^T X + X(A + \Delta A) = -(Q + \Delta Q), (Q + \Delta Q) \in S^+(n) \tag{10}$$

for a given $(Q_0 + \Delta Q_0) \in S^+(n)$ we can obtain:

$$X_P = g(A + \Delta A, Q_0 + \Delta Q_0) \in S^+(n). \tag{11}$$

In a similar way the solution set of the perturbed equation (10) is the following $\Psi_P := \{g(A + \Delta A, Q + \Delta Q) \in S^+(n) : (10)\}$. Analogically we can state that an admissible solution set of the LMI (4) can be found using the solution set of the perturbed Lyapunov equation if the following condition is satisfied $(Q_0 + \Delta Q_0), (Q + \Delta Q) \in S^+(n)$, i.e.:

$$\Omega_P := \{g(A + \Delta A, Q + \Delta Q) : (Q + \Delta Q) \in S^+(n)\} \tag{12}$$

Further in the paper we will continue with linear sensitivity analysis of the LMI (3) having in mind the LMI (4). An admissible solution of $X_0 \in S^+(n)$ of the LMI (3) can be obtained by applying the nominal solution of the Lyapunov equation (7) for $Q_0 \in S^+(n)$. Let us consider the perturbed Lyapunov equation (10) since for particular matrices Q_0 and ΔQ_0it can be written in the following way:

$$A^T X_0 + X_0 A + A^T \Delta X_0 + \Delta X_0 A + \Delta A^T X_0 + X_0 \Delta A + N(\Delta A, \Delta X_0) = -(Q_0 + \Delta Q_0) \tag{13}$$

where the terms of second order of Δ and ΔX_0 are put into the term $N(\Delta A, \Delta X_0)$. Further in the paper this term will be eliminated since we perform a linear sensitivity analysis. Using the equation (7) we obtain the expression

$$A^T \Delta X_0 + \Delta X_0 A + \Delta A^T X_0 + X_0 \Delta A = -\Delta Q_0. \tag{14}$$

We perform the set up

$$\Delta A^T X_0 + X_0 \Delta A = \Delta \tilde{Q}. \tag{15}$$

In order to obtain the equality

$$A^T \Delta X_0 + \Delta X_0 A = -\Delta Q_0 - \Delta \tilde{Q}. \tag{16}$$

We use the Kronecker product to rewrite the equation (15) in a matrix-vector form:

$$T_t \Delta a = \Delta \tilde{q} \tag{17}$$

here we use the expressions

$$T_t = I_n \otimes X_0 + (X_0 \otimes I_n)\Pi,$$

$$\Delta a = vec(\Delta A), \Delta\tilde{q} = vec(\Delta\tilde{Q}) \tag{18}$$

and Π is a permutation matrix. Equation (17) states the relation:

$$||\Delta\tilde{q}||_2 \leq ||T_t||_2||\Delta a||_2. \tag{19}$$

In a similar way the equation (16) can be transformed into:

$$T\Delta x_0 = -\Delta q_0 - T_{tn}\frac{\Delta a}{||A||_2}, \tag{20}$$

where

$$T = I_n \otimes A^T + A^T \otimes I_n, ||T_{tn}||_2 = ||T_t||_2||A||_2,$$

$$\Delta q_0 = vec(\Delta Q_0), \Delta x_0 = vec(\Delta X_0). \tag{21}$$

Since matrix A is a stable matrix then the Lyapunov operator is invertible, i.e. the matrix T is invertible. In such a way the relative perturbation bound of the admissible solution of the LMI (3) is shown below

$$\frac{||\Delta x_0||_2}{||X_0||_2} \leq \frac{1}{||X_0||_2}\left(||T_1||_2\frac{||\Delta q_0||_2}{||Q_0||_2} + ||T^{-1}||_2||T_{tn}||_2\frac{||\Delta a||_2}{||A||_2}\right), \tag{22}$$

where $||T_1||_2 = ||T^{-1}||_2||Q_0||_2$.

4 Numerical examples

We perform sensitivity analysis of the LMI (3).

Example 1. Consider the following matrices $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ and $Q_0 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, we apply the perturbations $\Delta = 10^{-i}A$ and $\Delta Q_0 = 10^{-i}Q_0$, where $i = 8, 7, \ldots, 4$. First we calculate the relative real perturbation $\frac{||X_P - X_0||_2}{||X_0||_2}$ in the admissible solution of the LMI (3) after applying the considered perturbations. Afterwards we compute the relative perturbation bound of the admissible solution of the LMI (3) according to expression (22). The obtained results are put in the table shown below:

Table 1: Sensitivity analysis results for Example 1

i	$\frac{\|\|X_P - X_0\|\|_2}{\|\|X_0\|\|_2}$	Bound (22)
8	$1.5*10^{-8}$	$3.5*10^{-8}$
7	$1.5*10^{-7}$	$3.5*10^{-7}$
6	$1.5*10^{-6}$	$3.5*10^{-6}$
5	$1.5*10^{-5}$	$3.5*10^{-5}$
4	$1.5*10^{-4}$	$3.5*10^{-4}$

Example 2. Consider the following matrices $A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ and $Q_0 = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix}$, we apply the perturbations $\Delta = 10^{-i}A$ and $\Delta Q_0 = 10^{-i}Q_0$, where $i = 8, 7, \ldots, 4$.

The obtained results are put in the table shown below:

Using the suggested solution method to perform perturbation analysis of the continuous-time matrix Lyapunov inequality, we obtain the perturbation bound of the admissible solution (22). This bound is close to the real relative perturbation bound $\frac{||X_P - X_0||_2}{||X_0||_2}$, which means that they its good in sense that they its tight.

Table 2: Sensitivity analysis results for Example 2

i	$\frac{\|\|X_P - X_0\|\|_2}{\|\|X_0\|\|_2}$	Bound (22)
8	$1.4*10^{-8}$	$3.8*10^{-8}$
7	$1.4*10^{-7}$	$3.8*10^{-7}$
6	$1.4*10^{-6}$	$3.8*10^{-6}$
5	$1.4*10^{-5}$	$3.8*10^{-5}$
4	$1.4*10^{-4}$	$3.8*10^{-4}$

5 Conclusion

In this paper we have proposed an approach to obtain an admissible solution set of the continuous-time matrix Lyapunov inequality using the solution set of the corresponding continuous-time Lyapunov equation. Tight linear perturbation bounds were computed for the set of admissible solutions of the continuous-time matrix Lyapunov inequality, which are linear functions of the data perturbations. Taking into account the obtained theoretical results we have presented numerical examples to vividly express the applicability and performance of the proposed solution approach to investigate the sensitivity of the matrix Lyapunov inequality for continuous-time systems.

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